1972

An Investigation Into the Transient Behavior of Multichannel Queueing Systems

James R. McCormick

129

AN INVESTIGATION INTO THE TRANSIENT BEHAVIOR OF MULTICHANNEL QUEUEING SYSTEMS

BY

JAMES R. McCORMICK

A thesis submitted
in partial fulfillment of the requirements for the
degree Master of Science, Major in
Mechanical Engineering,
South Dakota State University

1972

AN INVESTIGATION INTO THE TRANSIENT BEHAVIOR OF MULTICHANNEL QUEUEING SYSTEMS

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable for meeting the thesis requirements for this degree. Acceptance of this thesis does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser — Date

Head, Mechanical Engineering Department — Date

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Dr. Richard P. Covert for the guidance and counseling which led to the work presented here.

The author also wishes to thank Professor John Sandfort for his guidance throughout the author's graduate program.

J.R.M.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

GLOSSARY OF TERMS*

λ = Mean arrival rate

μ = Mean service rate per channel

$\rho = \lambda/m\mu$ (utilization factor)

P_n = Probability that 'n' units are in the system

m = Number of parallel service channels

---

*A partial listing of the definitions used in Queueing Theory appears in Appendix A

CHAPTER I

INTRODUCTION

In today's world, the waiting lines or queues formed by people, machines, or other units are common and occur whenever the temporary demand for a service exceeds the capacity to provide it. The absence or presence of a queue is of no consequence unless there is an economic value associated with it. These values may be the costs associated with idle production equipment and personnel or the lost sales due to unsatisfied customers. If one designs for sufficient facilities so that no queue builds up, then the operation is likely to be uneconomical due to excessive idle time at the service facility. If one designs the facility so that there is never any idle time at the service facility, then an unmanageable queue and many lost customers may result. A systems analyst must design a practical system so as to balance these costs at some optimal point.

Classical queueing problems may be found in many areas. In the transportation field, one must decide the optimal number of aircraft flights to schedule into a particular city to handle individual requests for freight and passenger service. In machine interference, when one operator is assigned to two or more machines, there is a possibility that one machine may require his services while he is working with the other. As the number of machines per operator is increased, the probability of one machine interfering with the normal operation of another is similarly increased. In a supermarket system, one must know the number of checkout counters to be provided. In a medical facility, one would want to know how many physicians should be assigned to an outpatient clinic.

Physically a queueing system is composed of two parts: a waiting line and a service facility. A complete description of a particular queueing system is dependent on the input process, service mechanism, queue discipline, station configuration, and population. Some of the variations for these parameters that have been solved mathematically are shown in Figure 1.[1]

Another way to designate queueing problems is by consideration of their properties. Thus, problem areas

---

[1]Richard P. Covert, <u>Queues and Simulation</u> (Brookings: Department of Mechanical Engineering, South Dakota State University, 1968), p. 9.

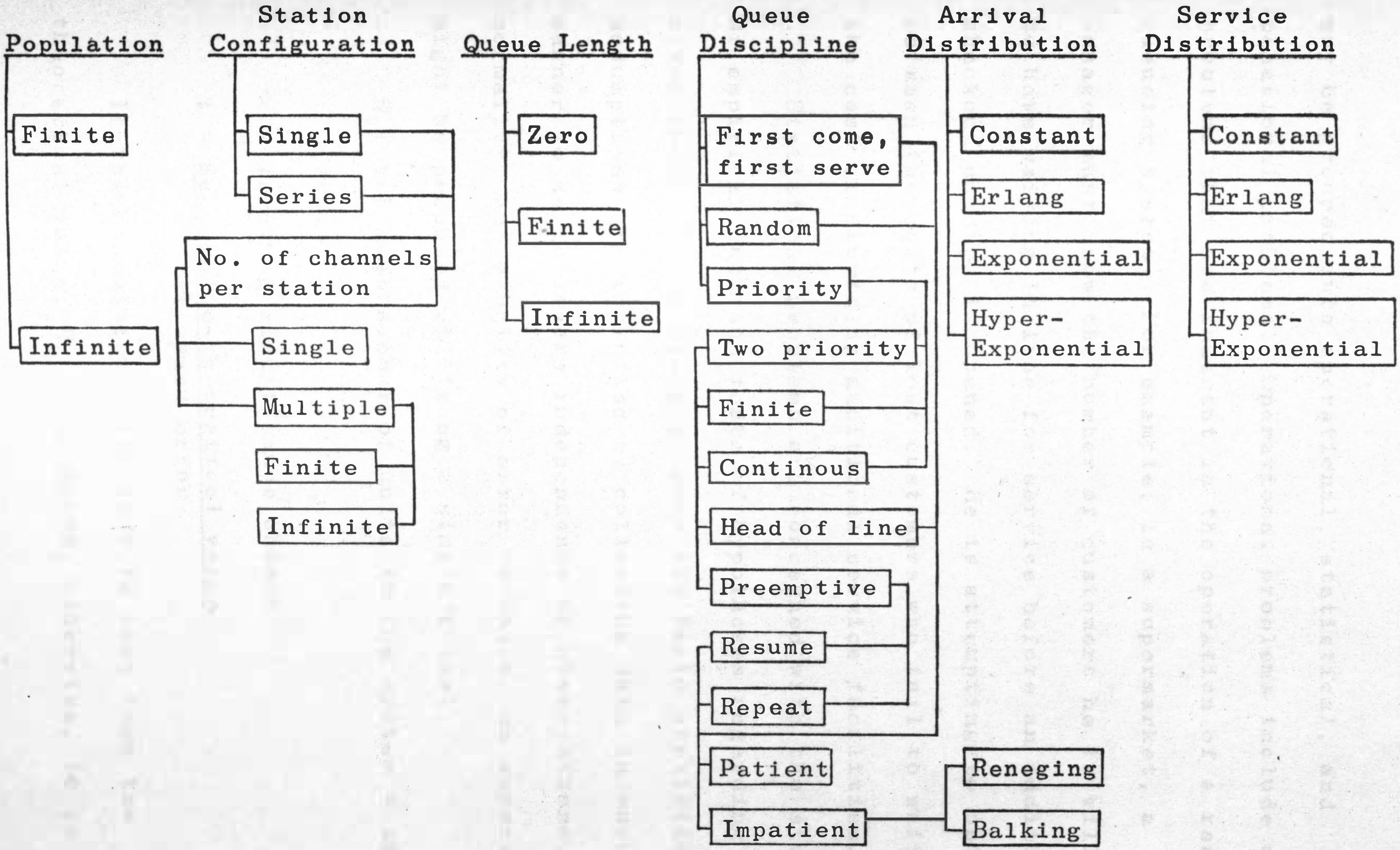


Figure 1. Variables in Queueing Situations

may be grouped into operational, statistical, and behavioral problems. Operational problems include all problems that are inherent in the operation of a real queueing system. For example, in a supermarket, a manager must know the number of customers he is willing to have waiting in line for service before an additional checkout counter is opened. He is attempting to optimize between the costs of lost customers who fail to wait and the cost of providing additional service facilities.

Statistical problems are concerned with the study of empirical data and tests of hypotheses regarding queue situations. As an example, once the basic statistical assumptions are satisfied by collecting data in such a manner so as to assure independence of observations, normality, and equality of error variance; an experiment might be proposed utilizing a simple 't' test.

H_o: the mean number of units in the system = theoretical value

α = level of significance chosen

$$t = \frac{\text{mean value-theoretical value}}{\text{standard error}}$$

If the calculated value of 't' is less than the theoretical value, H_o is accepted; otherwise, it is rejected.

The behavioral problems are concerned with mean queue length, mean number of units in the system, average waiting time, and similar statistics which describe the behavior of the system. Another behavioral distinction that could be made is between time-dependent or transient, and time-independent or steady-state behavior.[2]

A queueing process is defined to be in statistical equilibrium or steady state if the probability that there are n units in the system is independent of the time that has elapsed since the process started. That is:

$$P_n(t) = P_n.$$ [3]

For example, the average number of customers in a supermarket checkout queue will be different 10 minutes after the store opens from the average number 2 hours later. At each time there is a probability distribution for the length of the queue or the number in the system. The resulting probability distributions will vary with time. As time progresses, the mean number in the system moves away from the initial number in the system. This starting

---

[2]Narayan U. Bhat, "Sixty Years of Queueing Theory," Management Science, 15:6 (February, 1969), pp. B280-B294.

[3]Alec M. Lee, Applied Queueing Theory (London: MacMillan and Company, Ltd., 1966), p. 27.

point conditions the probabilities during the transient period. Later, the transient disappears and the system operates with the input and output of customers establishing distributions that are independent of the initial number present. When the initial effect has worn off, one might expect to find the system with the same general pattern of probabilities at any one time as at any other time. Thus, the probabilities acquire an equilibrium or a steady-state form.[4]

The majority of the literature in queueing theory is concerned with the behavioral aspect. The purpose of a study of behavioral problems is to understand a particular situation as thoroughly as possible. This is done through the use of mathematical models which are necessarily idealized to various degrees.

To construct formulas for mean behavioral statistics, one uses the differential-difference method developed by A. K. Erlang to model the queueing system mathematically. This method uses Markov-chain procedures.[5]

---

[4]Thomas L. Saaty, Elements of Queueing Theory (New York: McGraw-Hill Book Company, Inc., 1960), pp. 14-15.

[5]Joseph A. Panico, Queueing Theory: A Study of Waiting Lines for Business, Economics, and Science (Englewood Cliffs: Prentice-Hall, Inc., 1969), pp. 121-129.

The system of linear equations relates the stationary probability states of the transition matrix.[6] The steady state solution is obtained by first reducing the set of difference equations to a set of time-dependent equations. The time-rate-of-change is then set equal to zero, and the resulting equations are solved for the probability states. The probability states are then used to obtain the statistics desired.[7] These equations can be written without much difficulty for the simpler cases.

It is questionable whether in real-world processes such a thing as steady state exists. However, it is possible to make a great deal of progress in understanding the behavior of queues, whether they are time independent or not by using formulas for the steady.[8]

For the single-channel queueing system with exponential interarrival and service times, Morse derived a relationship for the relaxation time. This is defined as the time required for the system to reach 1/e of the difference

---

[6]Emanuel Parzen, Modern Probability Theory and Its Applications (New York: John Wiley and Sons, Inc., 1960), pp. 136-145.

[7]Covert, pp. 33-43.

[8]Lee, p. 215.

between its initial state and final steady-state value in terms of the mean steady-state arrival rate, λ, and the mean steady-state service rate, μ.* This can be used to plan for the length of transient periods in the single-channel case.[9]

If a system is entirely time dependent, then the system will never operate at steady-state conditions. This is a pure-transient system which is undesirable in real life and is primarily of academic interest. Applied researchers are generally interested in the effects of transient periods after which the systems operate at or near steady-state conditions. Lee suggested that when either the customers or servers or both are human beings, starting up periods are marked by such instability in service-time distributions and, frequently, interarrival-time distributions; that the fitting of a precise mathematical model becomes very difficult.[10]

Lee further suggests that the majority of queues

---

*A partial listing of the terms used appears in the Glossary of Terms and in Appendix A.

[9] P. M. Morse, Queues, Inventories, and Maintenance (New York: John Wiley and Sons, 1958), p. 67.

[10] Lee, p. 27.

involving people appear to operate at or near steady-state conditions either by design or by effects which tend to preserve stability. As an example, he recalled that at certain London railroad terminals it is possible to observe an old man who flags passing taxis into the feeder taxi rank when there are vacancies. By this means, the ranks are kept full and the old man is reimbursed in some small way for his services by the drivers. The system operates smoothly and apparently in steady state.[11]

In summary then, one finds that transient effects are present in almost all practical queueing situations. Starting conditions affect the transient state so that short term transients exist when the system starts and when changes in the system-operating parameters occur. Human beings often vary their behavior so that a particular system may appear to be operating in steady state. General guidelines for the length of transient periods do exist for single-channel systems.

---

[11]Lee, p. 216.

CHAPTER II

LITERATURE SEARCH

Explicit mathematical solutions for the transient single-channel queue have been proposed by several authors.

Morse derived a solution for the single-channel case with exponential interarrival and service times and found that the relaxation time is $1/(\sqrt{\mu}-\sqrt{\lambda})^2$. Relaxation time is the time required for the system to reach 1/e of the difference between its initial state and final steady-state value.[1]

For example, if $\lambda = 7$ per hour and $\mu = 10$ per hour, the relaxation time is equal to 3.78 hours. Nominal steady state is reached in 7.56 hours. This is shown in Figure 2 for the probability of being in state zero.

Although the probability of any state could be followed from time-zero to the steady-state conditions, it is convenient to use the probability of the station being idle since it is normally 1.0 at time-zero and will decrease to its steady-state value. In general the transient decays exponentially in time towards the steady-state conditions.

---

[1]Philip M. Morse, Queues, Inventories, and Maintenance (New York: John Wiley and Sons, 1958), p. 67.

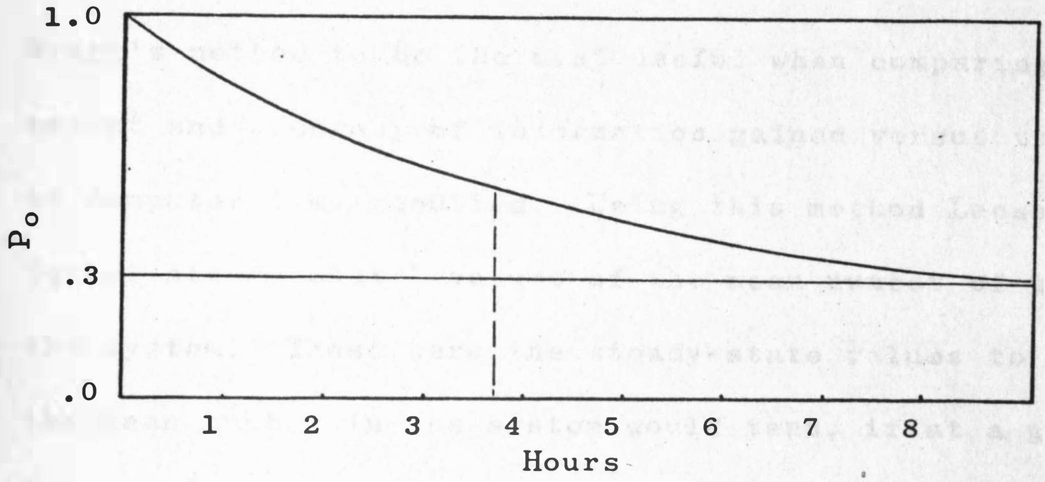


Figure 2. Probability of the System Being in State Zero at Various Points in Time after Operation Starts

Leese compared several numerical methods for determining transient behavior for the exponential single-channel queue with variable-arrival rate. The methods Leese compared were:

1. Direct numerical solution of the differential equations
2. Solution of the hyperbolic partial-differential equation for the generating function
3. Solution using a generalized-generating function
4. Taylors series computation of $P_n(t+h)$ from $P_n(t)$
5. Solution using a series of P_n in powers of ϱ
6. P_n expressed as a modified power series in ϱ
7. Wragg's method using an integral equation
8. Monte-Carlo simulation

In terms of the economy of calculation, Leese considered

Wragg's method to be the most useful when comparing the amount and accuracy of information gained versus the amount of computer time required. Using this method Leese computed "quasi-steady state" values of the mean number of units in the system. These were the steady-state values to which the mean number in the system would tend, if at a given time, t, the value of the utilization factor, ρ, were fixed at the value $\rho(t)$. Their results tended to verify Morse's equation for the relaxation time, in that after a short initial transient period, the expected number in the system follows the "quasi-steady state" values with time lags which were small when ρ was small and were much larger when ρ approaches unity.[2]

Bhat and Mann, as referenced by Bhat, studied the single-channel queue with Poisson arrivals and constant-service times using numerical methods to determine how long the system must be in operation before the steady state is reached. With a utilization factor of $\rho = .7$, they found that it takes the completion of 40 services after the start

---

[2]E. L. Leese, "Numerical Methods of Determining the Transient Behavior of Queues with Variable Arrival Rate," Queueing Theory: Recent Developments and Applications, ed. R. Cruon (New York: American Elsevier Publishing Company, Inc., 1967), pp. 86-97.

of the system to attain steady state. In an 8-hour work day with a service time of 6 or 7 minutes duration, it would take more than half the day to come to a state where equilibrium results would hold.[3]

Saaty derived a transient solution for the multichannel queue with a finite number of channels, c, in parallel and with an initial number of units, i, waiting for service. Saaty's final solution is in terms of the Laplace transform of a moment generating function.

For $n \geq c$

$$P_n^*(s) = \frac{\mu}{\lambda}\left[\sum_{j=0}^{c-1}(c-j)P_j^* \frac{\lambda}{c\mu\alpha_2^{n-j}} \cdot \frac{1-(\alpha_2/\alpha_1)^{n-j+1}}{1-(\alpha_2/\alpha_1)} - (c-j)P_j^*(s)\frac{\lambda}{c\mu\alpha_2^{n-j-1}} \cdot \frac{1-(\alpha_2/\alpha_1)^{n-j}}{1-(\alpha_2/\alpha_1)}\right] - \frac{1}{c\mu\alpha_2^{n-i-1}} \cdot \frac{1-(\alpha_2/\alpha_1)^{n-i}}{1-(\alpha_2/\alpha_1)}$$

where $_k = \dfrac{\lambda + c\mu + s \pm \sqrt{(\lambda + c\mu + s)^2 - 4c\mu\lambda}}{2\lambda}$ for $k = 1, 2$

n = number of units in the system

By expanding this solution in series, then taking successive derivatives and evaluating these derivatives about $s = 0$, the moments of the distribution of the mean number in the system can be obtained.[4]

---

[3]Narayan U. Bhat, "Sixty Years of Queueing Theory," Management Science, 15:6 (February, 1969), pp. B290-B291.

[4]Thomas L. Saaty, Elements of Queueing Theory (New York: McGraw-Hill Book Company, Inc., 1960), pp. 110-117.

Saaty's solution is very complex and the only information gained from the solution is an estimate of the distribution of the number of units in the system.

Consequently, Lee suggested that mathematical solutions to transient systems are either unobtainable or unmanageable.[5]

A solution may be obtained from a model either by mathematical analysis or by simulation. In this case strict mathematical analysis does not provide the information desired, and a simulation should be considered.

Simulation may be defined as the manipulation of the parameters of a fixed logical structure which models the real system in order to derive a solution pertinent to the real system. When one or more of the parameters of the system, namely the mean interarrival and service time, are stochastic variables whose individual values are randomly selected from a known probability distribution of possible values; then the simulation approach is called the Monte-Carlo technique.

Ackoff and Sasieni suggested the use of simulation models in the study of transitional processes. For example, the solution of a complex inventory problem involving the

---

[5]Alec M. Lee, <u>Applied Queueing Theory</u> (London: MacMillan and Company, Ltd., 1966), p. 27.

purchasing, storage, and use of spare parts for aircraft may show that the current stock levels for some items are too high while others are too low. An analytical solution may show only the average inventory level after a transition period. During the transition period those levels that were low are easily brought up by buying parts; however, those stock levels that were too high are depleted only through use. Although in time the inventory level may decrease, during the transitional period it will usually increase. A simulation study will indicate how much the inventory level may increase and how long a period is involved before the steady state is reached.[6]

Leese used Monte-Carlo simulation as one method of solution for the transient single-channel case with variable-arrival rate. Figure 3 is the input data used by Leese in his comparisons.

The results of 100 runs of a Fortran Program for the mean number in the queue, Q, are noted as crosses in Figure 4. The results of 100 runs of a program written in Simscript are noted as circles in Figure 4. Leese felt that

---

[6]Russell L. Ackoff and Maurice W. Sasieni, Fundamentals of Operations Research (New York: John Wiley and Sons, Inc., 1968), p. 108.

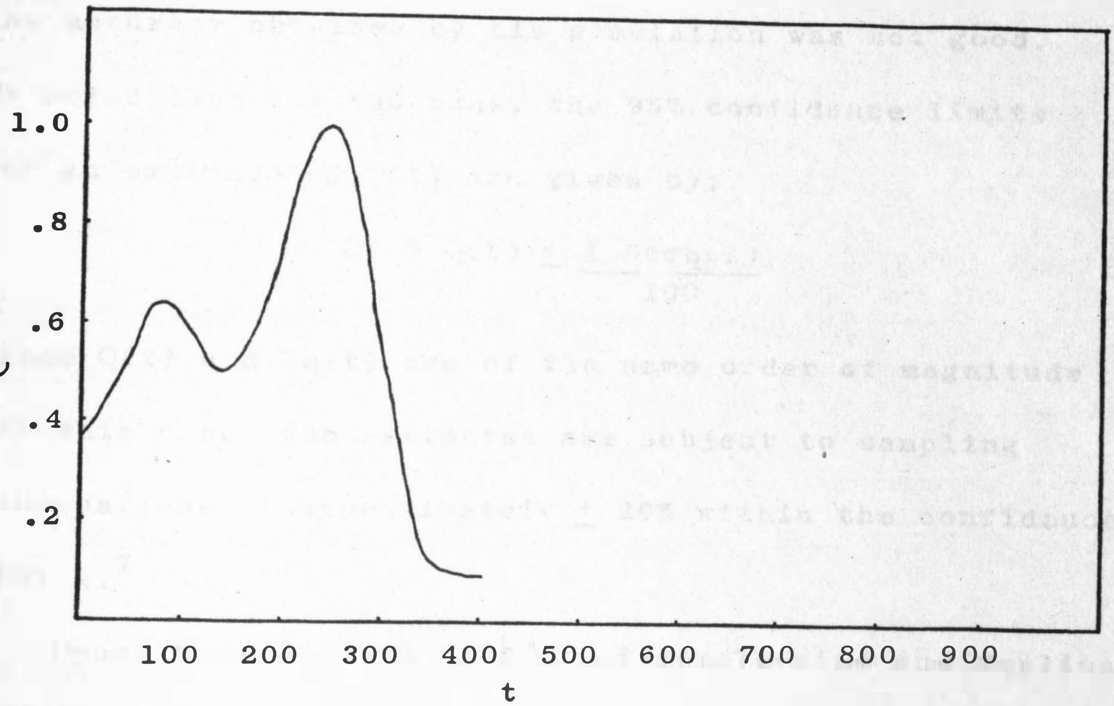


Figure 3. Mean Arrival Rate and Time Input Data as Used by Leese

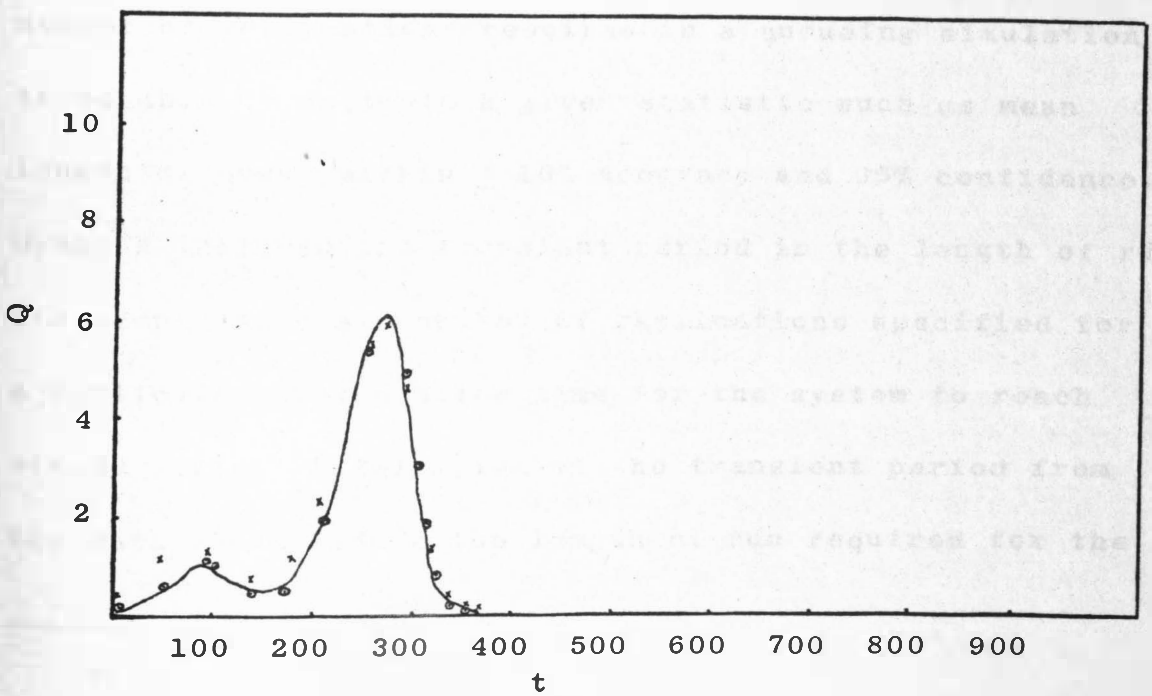


Figure 4. Average Number of Units in the Queue and Time From Simulation Studies Used by Leese

the accuracy obtained by his simulation was not good. He noted that for 100 runs, the 95% confidence limits for an estimate of Q(t) are given by:

$$CL = Q(t) \pm \frac{1.96\sigma_q(t)}{100}$$

Since Q(t) and $\sigma_q(t)$ are of the same order of magnitude for this case, the estimates are subject to sampling fluctuations of approximately ± 20% within the confidence limits.[7]

Dyanesh studied the effect of sample-size and replication on the accuracy of queueing simulations. The purpose of his study was to specify the minimum sample-size and number of replications required in a queueing simulation to be able to estimate a given statistic such as mean length of queue within ± 10% accuracy and 95% confidence. Dyanesh included the transient period in the length of run. The sample-size and number of replications specified for a particular system allow time for the system to reach steady state. Elimination of the transient period from the data would reduce the length of run required for the

---

[7]Leese, p. 96

same level of accuracy.[8]

As was shown earlier, Morse's mathematical solution for the transient single-channel case yielded a relationship between the mean steady-state arrival rate, λ, and service rate, μ, which gives a measure of the length of the transient period. However, for the multichannel case, one notes that Saaty's mathematical solution yields only moments of the distribution of the mean number of units in the system.

No guidelines for the length of transients in multichannel queueing systems were found in the literature. This information should be valuable to systems analysts who have a need for error analysis and for understanding the general behavior of transients in multichannel systems.

Therefore, it is proposed to study the lengths of initial transients in various multichannel situations by means of a digital-computer simulation. Then, by observing the probability states as they approach steady state, one should be able to gain a measure of the length of operation required for the system to reach steady-state conditions.

---

[8]Musvathy K. Dyanesh, "The Effect of Replication on Queueing Simulations" (unpublished Master's thesis, South Dakota State University, 1969), p 48.

CHAPTER III

MODEL AND PROCEDURE

The purpose of this experiment is to investigate the length of initial transient periods in multichannel queueing systems which have exponentially distributed interarrival and service times. The objective is to establish some guidelines for the length of this period for various system configurations and operating parameters.

In order to observe the states of the system, a simulation model must have two basic requirements for data output. The data output must include:

1. The number of units in the system
2. A time value associated with the number of units in the system

A sampling method which fits this requirement and also insures independence of observations is the time-slicing method in which the system is sampled at predetermined time intervals and appropriate data gathered. Since the parameters of the system λ and μ, are means of random variables, sampling the system at fixed time periods is in effect a random sampling procedure; and independence of observations can be assumed.

Programs for digital computer simulations of queueing systems may be written in compiler languages such as Fortran, Algol, and PL/1 or in special purpose macro languages which are designed for ease of programming simulations. These macro languages such as GPSS, SIMSCRIPT, GASP, SIMPAC, DYNAMO, and SIMULATE are not currently available for use on the local IBM 360/30 computer.

Of the several simulation programs written in compiler languages, a program written by Gould in Fortran IV was the most suitable.[1] It was used with minor modifications which suppressed all output statistics and data except the sample time and the number of units in the system at the sample time. The capabilities of this program are shown in Table I. The program capabilities utilized in the experiment are enclosed by rectangles in Table I.

Queues which operate at or near steady-state conditions must by necessity have a utilization factor, ρ , defined as $\lambda/m\mu$, which is less than unity. If $\rho \geq 1$, steady-state conditions will not exist as the queue will increase without bound. For this experiment the value of ρ is varied between

---

[1]Richard P. Covert, Queues and Simulation (Brookings: Department of Mechanical Engineering, South Dakota State University, 1968), pp. 227-238.

Table 1. Computer Program Capabilities

Number	Parameter	Capability*
1	Number of parallel channels	$\boxed{10}$
2	Type of queue	a. Finite b. $\boxed{\text{Infinite}}$
3	Interarrival time distribution	a. $\boxed{\text{Exponential}}$ b. Constant c. Ten-celled histogram
4	Service time distribution	a. $\boxed{\text{Exponential}}$ b. Constant c. Ten-celled histogram
5	Queue discipline	$\boxed{\text{First in, first out}}$
6	Maximum number of units per run	$\boxed{999}$
7	Maximum number of replications	$\boxed{30}$

*The program capabilities utilized in the experiment are enclosed

.3 and .7 so that various load utilizations in the steady-state operating region may be observed. The number of channels, m, is varied within the program capabilities between 1 and 10 parallel channels. The combinations of the experiment are shown in Table 2.

Each experimental combination was simulated 10 times on an IBM 360/30 computer. The selection of 10 repetitions was arbitrary. Dyanesh indicated 30 repetitions for higher values of ρ in the estimation of cumulative statistics which necessarily include transient effects.[2] However, for this experiment one is interested only in observing what happens during the transient period. The length of the simulation was fixed to include a period after the transient effects had ended. The sampling period was fixed at .50 hour. Several trial runs showed that this interval gave satisfactory results and was reasonably optimal when the time required for printout and sampling efficiency were considered.

Gafarian and Ancker established a measure for the minimum sample time below which no gain in sampling efficiendy occurs. This was approximately equal to the

---

[2]Musvathy K. Dyanesh, "The Effect of Replication on Queueing Simulations" (unpublished Master's thesis, South Dakota State University, 1969), p. 48.

Table 2. Experimental Combinations

Number	ρ	λ^1	μ^2	m
1	.3	3	10.00	1
2		3	5.00	2
3		3	3.33	3
4		3	2.00	5
5		3	1.00	10
6	.5	5	10.00	1
7		5	5.00	2
8		5	3.33	3
9		5	2.00	5
10		5	1.00	10
11	.7	7	10.00	1
12		7	5.00	2
13		7	3.33	3
14		7	2.00	5
15		7	1.00	10

[1] λ is expressed as arrivals per hour
[2] μ is expressed as services per hour

interarrival time, $1/\lambda$.[3] For this case, where:

λ = 3 per hour $\quad$ $1/\lambda$ = .333 hour*

λ = 5 per hour $\quad$ $1/\lambda$ = .200 hour

λ = 7 per hour $\quad$ $1/\lambda$ = .143 hour

The sample time used was fairly close to the largest time predicted for maximum sampling efficiency.

The data collected was simply the number of units in the system at the sampling time. A moving average of size 10 was next computed for each repetition. The relative frequency of occurrence of the system state in question was used to establish a percentage probability of occurrence. This probability was then associated with the midpoint of the 10 values and the sample time corresponding to the midpoint. The moving average was used as a curve-smoothing technique.# A grand mean for each experimental combination was then found by averaging the moving-average values of

---

[3] A. V. Gafarian and C. J. Ancker, Jr., "Mean Value Estimation From Digital Computer Simulation," Operations Research, 14:1 (January, 1966), pp. 25-44.

*Note that although for this study the time units are in hours, any unit of time may be used as long as it is consistent.

#Note that one no longer has independent observations at this point.

the repetitions by corresponding time periods. These grand means were assumed to be normally distributed about a population mean which varies with time during the transient and is independent of time when the system settles to steady state. These grand means were then graphed for each experimental combination as a probability state versus time. Figure 5 shows a representative graph for ρ = .7 and m = 2.

This graph indicated that random variations are occurring in the simulation even though the mean system is operating in steady state. This random variation of means of

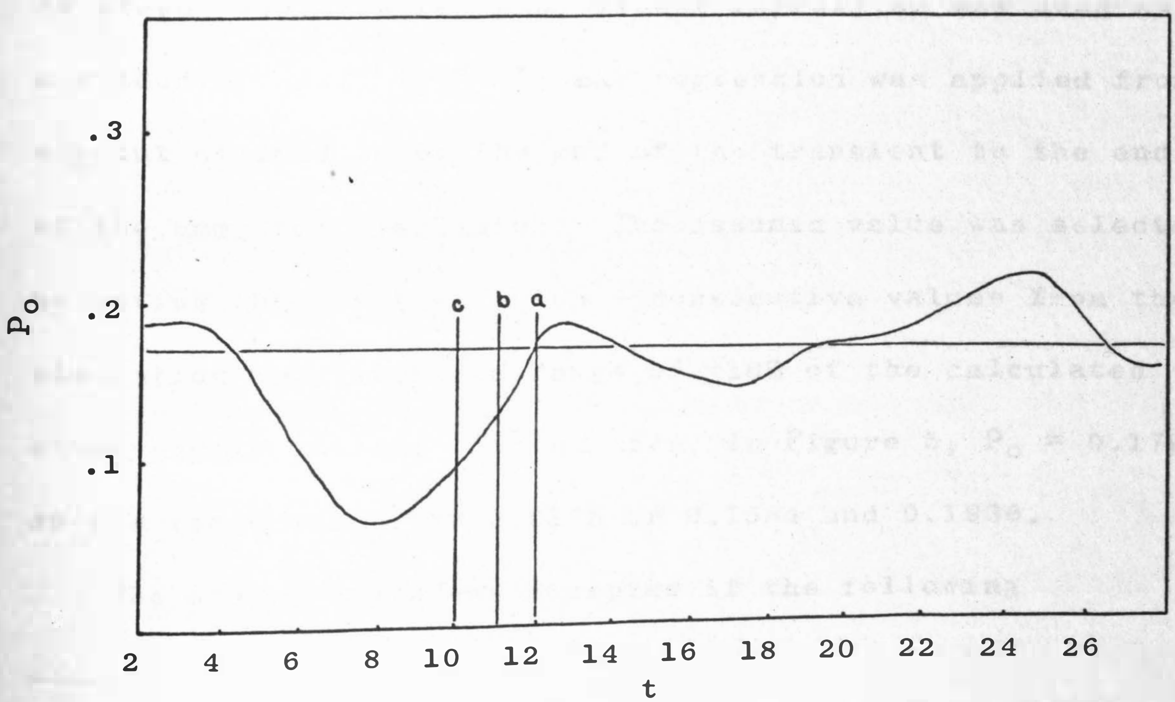


Figure 5. Probability of Being in State Zero and Time for a Mean of 10 Simulations with ρ = .7 and m = 2

individual states is to be expected as both the interarrival and service times are random variables. Therefore, the simulation probability states may never actually equal theoretical steady-state results. A more reasonable approach with simulation results would appear to be the establishing of limits on the theoretical steady-state probability value within which the actual value could vary randomly and still be considered to be operating in steady state. Note that the steady-state probability value is a constant whose equation is a straight line with slope zero. Since the simulation results vary randomly about some constant value as steady state is reached, linear regression was used as a method of analysis.[4] Linear regression was applied from a point assumed to be the end of the transient to the end of the computer simulation. The assumed value was selected by noting the point at which 3 consecutive values from the simulation were within a range of $\pm 10\%$ of the calculated steady-state values. For example, in Figure 5, $P_o = 0.176$, so the range was 0.176 ± 0.0176 or 0.1584 and 0.1936.

The steady state was accepted if the following

---

[4]Robert G. P. Steel and James H. Torrie, Principles and Procedures of Statistics (New York: McGraw-Hill Book Company, Inc., 1960), pp. 161-175.

conditions were satisfied:

1. The slope of the regression line was 0.0 ± 0.005

2. The 95% confidence interval on the regression line associated with the endpoint included the theoretical steady-state value

If the steady state were accepted, the regression was repeated including the next time back until one or both of the conditions were violated. The last sample time at which both criteria were satisfied was defined for the purpose of this study as the point at which steady state begins.

In Figure 5, note that the transient period appears to be over at 12 hours. It was at this point that linear regression analysis was started. Sample calculations are shown in Appendix C.

For the period 12-27 hours, the following results were obtained:

$$\text{slope} = .0017$$

$$CL(P_o) = .1457,\ .1913$$

This is shown as point 'a' in Figure 5. Both conditions were satisfied so the analysis was shifted back into the transient area by one sample time period.

For the period 11-27 hours, the following results were obtained: slope = .0024

$$CL(P_o) = .13606, .18194$$

This is shown as point 'b' in Figure 5. Both conditions were still satisfied so the regression analysis was extended back an additional time period.

For the period 10-27 hours, the following results were obtained: slope = .0031

$$CL(P_o) = .12604, .17126$$

This is shown as point 'c' in Figure 5. Note that here the slope still fits within the limits but that the confidence limits no longer include the theoretical steady-state value of $P_o = 0.176$. For this experimental combination, 11 hours was the time at which steady-state conditions were defined to start.

Each experimental combination was analyzed in this manner, and the results were tabulated.

An analysis of variance was used to test the results for significant differences. The level of significance was fixed at the 10% level due to the extremely variable nature of the simulation results.

CHAPTER IV

RESULTS

Simulation runs were made on all experimental combinations. The moving averages were computed, and the means of the moving averages were calculated by sample time period and graphed. Table 3 shows the theoretical steady-state results for all experimental runs. These are the values to which the simulation results should converge after the transient period has passed.

Table 3. Theoretical Steady-State Probability Values for Each Experimental Combination

Number	ρ	m	Probability
1	.3	1	P_o = .700
2		2	.538
3		3	.403
4		5	.223
5		10	.050
6	.5	1	.500
7		2	.333
8		3	.210
9		5	.080
10		10	.007
11	.7	1	.300
12		2	.176
13		3	.095
14		5	.026
15		10	.001

The simulation results of all experimental combinations converged to their respective theoretical steady-state values except for the single-channel simulations at utilization factors of .5 and .7. No set of values from these simulations met the conditions specified. This behavior does occur in the study of random phenomena and indicates that more replications were needed for these two combinations; however, since Morse's equation already predicts relaxation time for the single-channel system, these two combinations were not analyzed.

The results of the linear regression analysis are shown in Table 4. Note that the time units to steady state may be interpreted as days, hours, minutes, or seconds dependent on the time units associated with the input parameters.

An analysis of variance was conducted on the relationship between the utilization factor and the number of parallel service channels. The level at which significance was chosen to be tested was the 10% level. This level was chosen because of the extremely variable nature of the simulation results.

The results of the analysis of variance are shown in Table 5.

Table 4. Experimental Results

Number		m	Time Units To Steady State	slope	Regression Results $\bar{y}$	$CL(P_o)$	P_o
1	.3	1	2	-.0035	.6084	.5761-.7281	.700
2		2	7	-.0038	.5427	.5331-.5903	.538
3		3	9	-.0026	.4121	.3833-.4877	.403
4		5	12	-.0044	.2187	.1929-.3105	.223
5		10	23	.0000	.2220	.1974-.2466	.224[1]
6	.5	1	-	-	-	-	.500
7		2	5	-.0046	.3320	.3288-.4951	.333
8		3	8.5	-.0029	.2025	.2011-.2511	.210
9		5	10	-.0014	.0746	.0506-.0800	.080
10		10	35	.0000	.1928	.1600-.2256	.175[2]
11	.7	1	-	-	-	-	.300
12		2	11	.0024	.1782	.1360-.1819	.176
13		3	7	.0002	.0956	.0805-.1074	.095
14		5	8	.0007	.0290	.0121-.0327	.026
15		10	12	-.0006	.1312	.1182-.1532	.140[3]

[1] P_2 was used for analysis*
[2] P_4 was used for analysis
[3] P_6 was used for analysis

*P_2 is the probability of 2 units being in the system

Table 5. Analysis of Variance

Source	SS	df	ms	F*
Total	790.23	11	-	-
Utilization Factor	53.79	2	26.89	.673
Number of Channels	496.73	3	165.57	4.14*
Error	239.71	6	39.95	-

$^*F_{.10}\ (3/6) = 3.29$

Table 5 indicates that there were no significant differences between the levels of the utilization factor. Table 5 does indicate that there was a significant difference at the 10% level between the number of parallel channels.

A linear relationship between the time units required to reach steady state and the number of parallel channels is shown in Figure 6. Included in Figure 6 are the values for the single-channel case. These were calculated from Morse's equation for the relaxation time, T_r.[#] The regression equation for the relationship between time units

---

[#]The value shown is 2xTr

to steady state, T, and number of parallel channels, m, is:

$$T = 2.035m+2.12$$

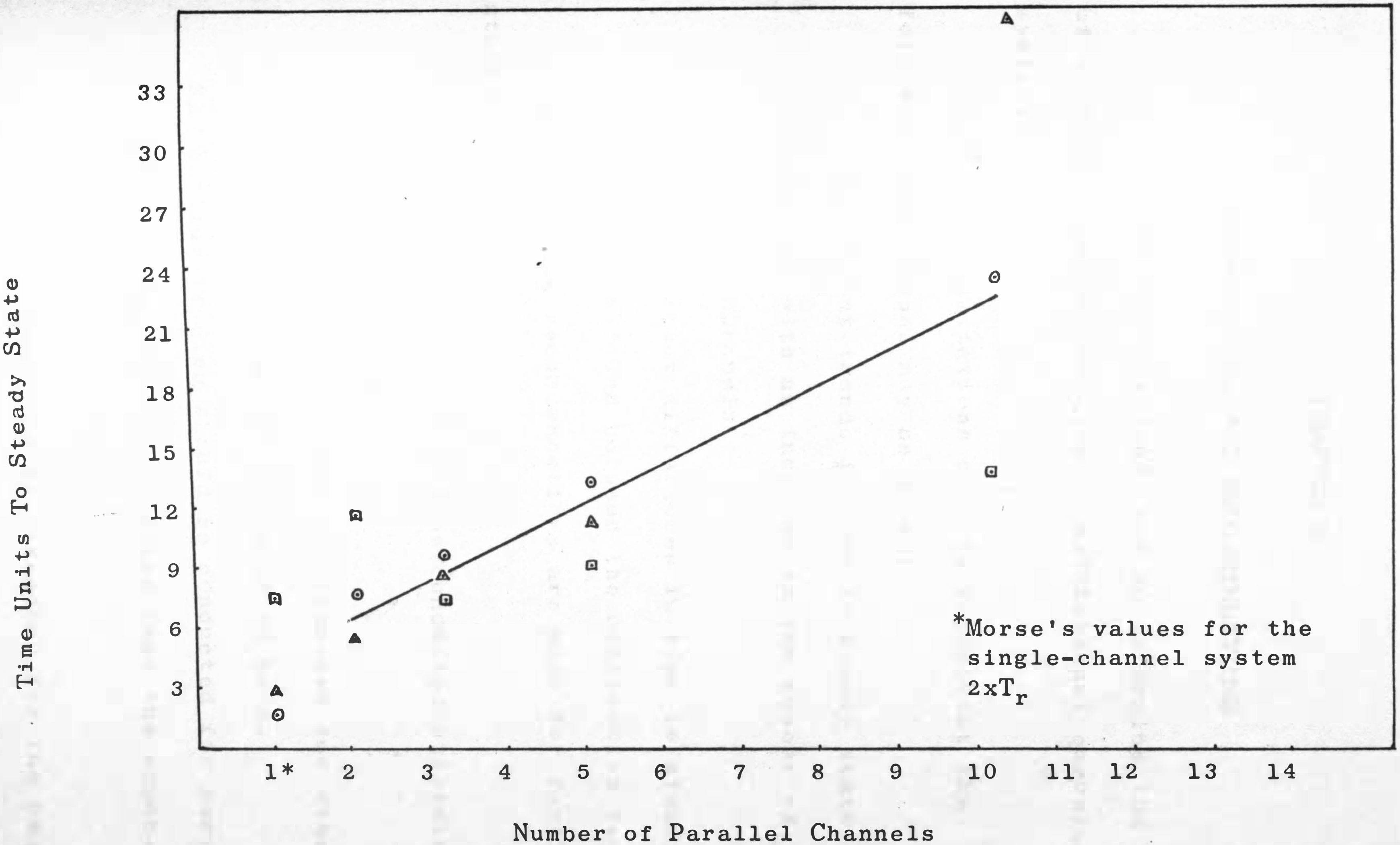


Figure 6. Time Units To Steady State and Number of Parallel Channels

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The purpose of this study was to determine the length of initial transient periods in multichannel queueing systems.

Within the limitations of this experiment the following conclusions may be drawn:

1. Significant trends in time to steady state occurred with an increase in the number of parallel channels.
2. No significant differences in time to steady state were found between the utilization factors.

The following recommendations are made for further study:

1. Similar studies should be conducted utilizing more repetitions.
2. Similar studies should be conducted for other utilization factors not studied here.
3. Similar studies should be conducted for service and interarrival times other than the exponential cases.
4. There appears to be no precedent for the use of

linear regression analysis in studies of this type. Similar studies should be conducted in this area to prove or disprove its merits relative to other testing techniques in the study of transients.

BIBLIOGRAPHY

Ackoff, Russell L. and Sasieni, Maurice W. Fundamentals of Operations Research, New York: John Wiley and Sons, Inc., 1968.

Bhat, Narayan U. "Sixty Years of Queueing Theory," Management Science, 15:6 (February, 1969), pp. B280-B294.

Covert, R. P. Queues and Simulations. Brookings, Department of Mechanical Engineering, South Dakota State University, 1968.

Cruon, R. (ed.) Queueing Theory, Recent Developments and Applications. New York: American Elsevier Publishing Company, Inc., 1967.

Dyanesh, Musvathy K. "The Effect of Replication on Queueing Simulations," Unpublished Master's thesis, South Dakota State University, 1968.

Gafarian, A. V. and Ancker, Jr., C. J. "Mean Value Estimation from Digital Computer Simulation," Operations Research, 14:1 (January, 1966), pp. 25-44.

Lee, Alec M. Applied Queueing Theory. London: MacMillan and Company, Ltd., 1966.

Morse, P. M. Queues, Inventories and Maintenance. New York: John Wiley and Sons, Inc., 1957.

Panico, Joseph A. Queueing Theory: A Study of Waiting Lines for Business, Economics, and Science. Englewood Cliffs: Prentice-Hall, Inc., 1969.

Parzen, Emanuel. Modern Probability Theory and Its Applications. New York: John Wiley and Sons, Inc., 1960.

Saaty, T. L. Elements of Queueing Theory. New York: McGraw-Hill Book Company, Inc., 1960.

Steel, Robert G. P. and Torrie, James H. Principles and Procedures of Statistics. New York: McGraw-Hill Book Company, Inc., 1960.

APPENDIX A

A Partial List of the Terms used in Queueing Theory and Simulation[1]

Event. Any noteworthy occurrence in the system is termed as an event. It may be an arrival or the completion of a service.

Facility. A facility is a generalized service station and may be used to indicate a complex of service stations.

Mean Arrival Rate. The mean arrival rate is the inverse of the mean time between arrivals and may have any type of a distribution.

Mean Service Rate. The mean service rate is the inverse of the mean time for service and may have any type of distribution.

Parameters. A parameter is a factor which describes a statistical population. These factors include the mean arrival rate and the mean service rate.

Queue. The queue consists of all units waiting to be

---

[1]Richard P. Covert, Queues and Simulation (Brookings: Department of Mechanical Engineering, South Dakota State University, 1968), pp. 2-8.

serviced at a specified service station.

<u>Service Station</u>. The performance of all services takes place in a service station.

<u>System</u>. The system is composed of a service facility and its queues.

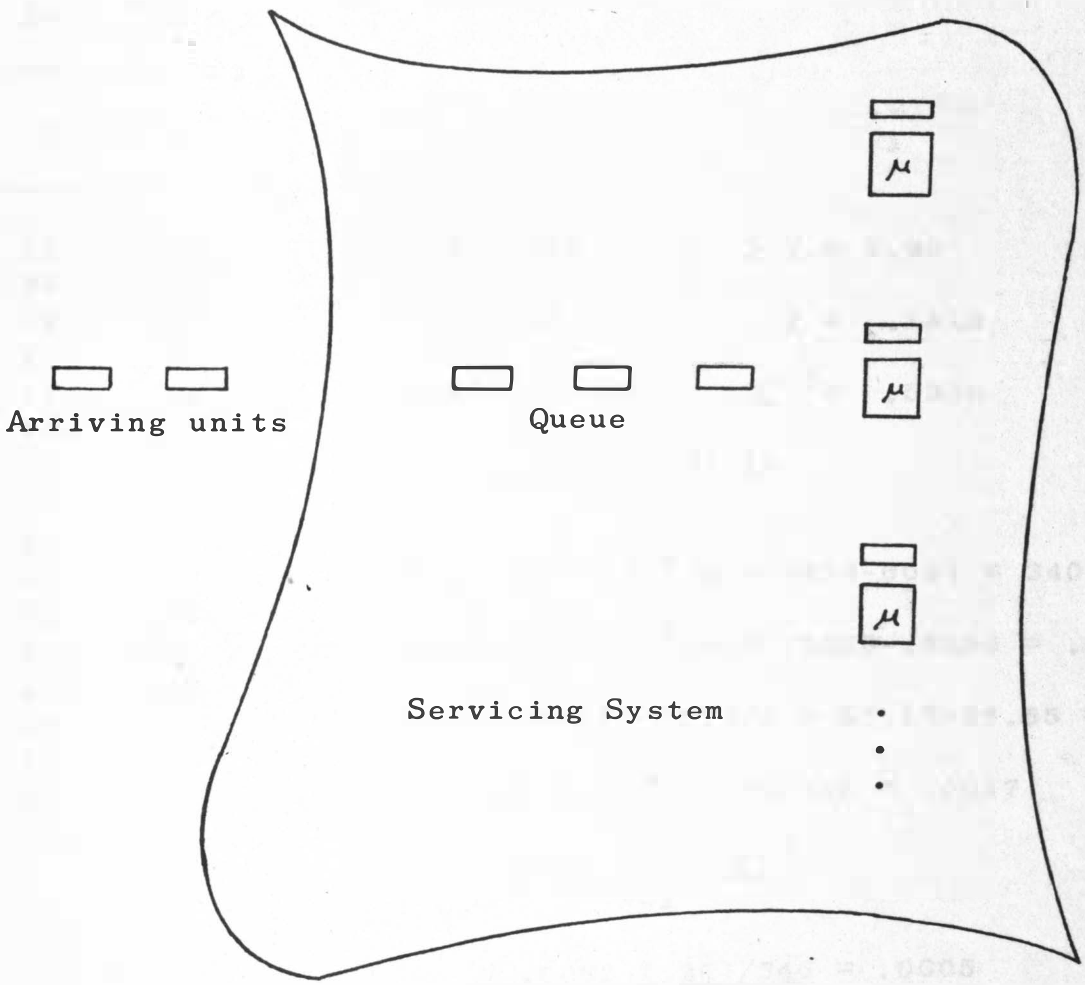
<u>System State</u>. The state of a system is the number of units in the system.

<u>Unit</u>. The unit is that which is to be serviced.

<u>Utilization Factor</u>. The ratio of the system mean arrival rate to the system mean service rate is termed the utilization factor.

APPENDIX B

Generalized Schematic Diagram of a
Multichannel System



APPENDIX C

Sample Calculations

Regression analysis of the simulation results from a two channel system with a utilization factor $\rho = .7$.

For the period 12-27 hours:

Data Table

X	Y
12	.19
13	.19
14	.17
15	.16
16	.16
17	.14
18	.16
19	.18
20	.17
21	.19
22	.20
23	.22
24	.22
25	.21
26	.18
27	.16

$\Sigma X = 312$ $\qquad \Sigma Y = 2.90$

$\bar{x} = 19.5$ $\qquad \bar{y} = .1812$

$\Sigma X^2 = 6424$ $\qquad \Sigma Y^2 = .5338$

$\Sigma XY = 57.15$

$$\Sigma x^2 = \Sigma X^2 - (\Sigma X)^2/n = 6424-6084 = 340$$

$$\Sigma y^2 = \Sigma Y^2 - (\Sigma Y)^2/n = .5338-.5256 = .0082$$

$$\Sigma xy = \Sigma XY - (\Sigma X)(\Sigma Y)/n = 57.15-56.55 = .60$$

$$b = \Sigma xy / \Sigma x^2 = .60/340 = .0017$$

$$S^2yx = \frac{\Sigma y^2 - (\Sigma xy)^2/\Sigma x^2}{n-2}$$

$$= \frac{.0082-(.36)/340}{14} = .0005$$

$$S_{yx} = .0224$$

$$CL(P_o)_{12} = \bar{y} + b(X-\bar{x}) \pm t_{.05}\, S_{yx}\sqrt{\frac{1}{n} + \frac{(X-\bar{x})^2}{\Sigma x^2}}$$

't' at n-2 df.

$$CL(P_o)_{12} = .1812 + .0017(12-19.5) \pm 2.145(.0224)\sqrt{\frac{1}{16} + \frac{(12-19.5)^2}{340}}$$

$$= .1457,\ .1913$$

Sample calculations for relaxation times of single channel systems using Morse's equation.

For: $\rho = .3$ $\quad T_r = \dfrac{1}{(\sqrt{10}-\sqrt{3})^2} = .488$ hours

$\rho = .5$ $\quad T_r = \dfrac{1}{(\sqrt{10}-\sqrt{5})^2} = 1.165$ hours

$\rho = .7$ $\quad T_r = \dfrac{1}{(\sqrt{10}-\sqrt{7})^2} = 3.78$ hours